

Lyapunov exponents from node-counting arguments

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Abstract

A conjecture connecting Lyapunov exponents of coupled map lattices and the node theorem is presented. It is based on the analogy between the linear stability analysis of extended chaotic states and the Schrödinger problem for a particle in a disordered potential. As a consequence, we propose an alternative method to compute the Lyapunov spectrum. The implications on the foundation of the recently proposed “chronotopic approach” are also discussed.

1 INTRODUCTION

In a series of recent papers [1, 2, 3], the so-called *chronotopic approach* has been developed with the aim of extending the by-now standard concept of Lyapunov spectrum [4] to spatially inhomogeneous perturbations. In 1d extended systems, the study of infinitesimal perturbations with an exponential profile (and a generic decay-rate μ) has led to introduce the generalized integrated density $n_\lambda(\lambda, \mu)$, defined as the fraction of Lyapunov exponents smaller than λ . In these notations, the usual spectrum is denoted by $n_\lambda(\lambda, 0)$, as it is characterized by the absence of an overall exponential spatial profile. In a complementary way, the spatial dynamics of perturbations has been studied by assigning a temporal growth rate λ and thereby determining μ , by following the evolution in tangent space along the space axis. In this case, one determines the spectrum of spatial exponents $n_\mu(\lambda, \mu)$. The main result of the chronotopic approach is the existence of a dynamical invariant, the *entropy potential* Φ , the knowledge of which allows to determine all properties of the evolution of localized as well as extended perturbations. However, the existence of Φ has been proved only in a few, very simple, cases of uniform space-time patterns and numerically tested just in some more realistic examples. No general argument has yet been found to justify the validity of the whole approach in generic 1d systems. The available “proof” (see in particular Ref. [2]) is essentially based on the observation that Lyapunov vectors (i.e. the eigenvectors of the stability matrix) are ordered from the most to the least unstable one (or viceversa) for increasing wavenumber k . Accordingly, one can describe the spatial structure of a generic Lyapunov vector with a single complex number $\tilde{\mu} = \mu + ik$, the real part of which is the exponential growth rate, while the imaginary part is nothing but the wavenumber or, equivalently, the integrated density $n_\lambda(\lambda, \mu)$. Analogously, a temporal frequency ω has been invoked to order all spatial Lyapunov exponents and thus to represent a measure of $n_\mu(\lambda, \mu)$. The frequency ω can be read as the imaginary part of the complex number $\tilde{\lambda} = \omega + i\lambda$, where λ is the temporal growth rate (i.e. the Lyapunov exponent) of the given perturbation. The analyticity properties of the “dispersion relation” connecting $\tilde{\mu}$ with $\tilde{\lambda}$ furnish the last ingredient to “prove” the existence of an

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entropy potential [2]. The key question is how far can one go with this type of arguments to prove the validity of the chronotopic approach? If we look at one particular consequence of the existence of Φ , namely that the Kolmogorov-Sinai entropy density h_{KS} is independent of the direction along which a 2d space-time pattern is thought to be generated, then we are led to expect a rather general validity. In fact, it looks rather plausible that h_{KS} is an intrinsic property of a given pattern, independent of the way we look at it!

In this paper we explore the possibility to introduce a general but meaningful definition of the wavenumber and, in turn, to define “rotation numbers” as the imaginary counterpart of the Lyapunov exponents. As a result, we propose an alternative method to compute the Lyapunov spectrum by using the transfer matrix approach rather than iterating the linear relations in time as it is usually done. The approach is limited to a class of coupled map lattices (CMLs) with everywhere expanding multipliers. Accordingly, it is not yet demonstrated that the perspectives so far outlined are consistent in general. Nevertheless, our results provide encouraging indications for future investigations.

The outline of the paper is as follows. In section 2, we discuss the simple case of fixed points (in time). In section 3, the approach is extended to periodic and chaotic patterns, introducing a conjecture and numerically testing it. In the last section, we discuss the limitations as well as possible further extensions.

2 ANALOGY WITH THE SCHRÖDINGER PROBLEM

The computation of both temporal and spatial Lyapunov spectra is normally carried out by resorting to the standard orthonormalization procedure (SOP) introduced many years ago [5]. In this section, we discuss the analogy between the linear stability analysis of CMLs and the 1d Schrödinger problem (see also [6]), with the aim of both strengthening the internal consistency of the chronotopic approach and to introduce the first elements of an alternative algorithm.

Let us consider a CML [7] and denote with x_n^i the field variable at lattice location i ($i = 1, \dots, L$) and time n . By introducing the L -dimensional column vector δX_n of the perturbations δx_n^i , we can synthetically express the evolution equations in the tangent space as

$$\delta X_{n+1} = M_n D_\varepsilon \delta X_n \quad , \quad (1)$$

where M_n is a diagonal matrix whose diagonal elements m_n^i are the derivatives of the (here unspecified) nonlinear mapping, and D_ε is the tridiagonal matrix associated to the diffusive coupling

$$(D_\varepsilon)_{i,j} = \frac{\varepsilon}{2} \delta_{i+1,j} + (1 - \varepsilon) \delta_{i,j} + \frac{\varepsilon}{2} \delta_{i-1,j} \quad . \quad (2)$$

A simple but instructive example that allows discussing the main ideas is that of frozen random patterns, for which m_n^i depends only on the spatial variable i . In this case, the estimation of the Lyapunov exponents λ reduces to the eigenvalue problem for the matrix MD_ε , namely

$$\Lambda \delta x^i = m^i \left[\frac{\varepsilon}{2} \delta x^{i-1} + (1 - \varepsilon) \delta x^i + \frac{\varepsilon}{2} \delta x^{i+1} \right] \quad (3)$$

where $\lambda = \log |\Lambda|$.

Eq. (3) resembles the tight-binding approximation of the 1d Schrödinger equation (with imaginary time) in the presence of a random potential V^i , namely the celebrated Anderson model

$$\omega \psi^i = \psi^{i+1} + \psi^{i-1} + (V^i - 2) \psi^i \quad , \quad (4)$$

where the eigenvalue ω plays the role of the multiplier Λ , while the eigenfunction corresponds to the Lyapunov vector of the CML. Accordingly, finding the spectrum of the Schrödinger operator is equivalent to finding the Lyapunov spectrum of the CML [8]. Incidentally, notice that an even closer analogy exists with the computation of the vibrational spectrum of a chain with random masses [9].

The spectrum (or the density of states) of the Schrödinger problem can be determined without actually diagonalizing the operator implicitly defined by the r.h.s. of Eq. (4) (which is just the sum of the discretized Laplacian and a diagonal operator). Indeed, its symmetry ensures the validity of the node theorem which states that the eigenfunctions are ordered according to the number of their zeros [10]. Accordingly, the structure of a given eigenfunction suffices to determine the position of the corresponding eigenvalue inside the energy spectrum. This task is usually accomplished by the transfer matrix approach.

It is natural to ask whether n_λ in a CML can be analogously determined from the spatial structure of the corresponding Lyapunov vector. This question can be easily answered in the positive for frozen random patterns. In fact, although the operator in the r.h.s. of Eq. (3) is not symmetric, one can easily realize that the change of variables $\delta x^i \rightarrow \sqrt{m^i} \delta x^i$, leads to a fully symmetric structure. As a consequence, the node theorem applies also in this restricted CML problem. Furthermore, the symmetry of the operator guarantees that the eigenvalues are all real, i.e. no rotations in tangent space are involved.

In practice, it is sufficient to proceed as follows: one starts fixing $\Lambda = \exp(\lambda)$ in Eq. (3), where λ is the Lyapunov exponent of interest. We then iterate Eq. (3) along a spatial direction (left and right are equivalent directions) and count the number of changes of sign of δx^i from one to the next site. The fraction of such changes (i.e. the number of zeros per unit length) equals the integrated density of Lyapunov exponents n_λ . In Fig. 1 we compare the results of such a procedure with the SOP. The perfect agreement represents a direct verification of the validity of the approach.

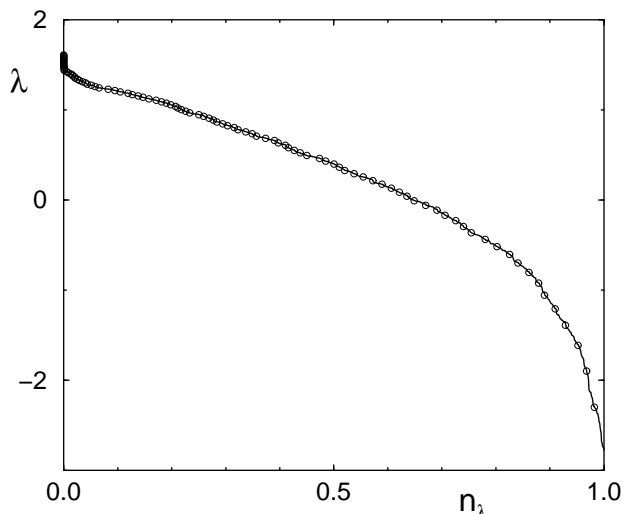


Figure 1: Lyapunov spectrum for a frozen random pattern. The solid curve refers to the result of the SOP, while open circles correspond to the outcome of the node counting approach. Here and in all subsequent cases, we have always considered $\varepsilon = 1/3$.

Let us notice that, from the point of view of the chronotopic approach, this way of determining n_λ is very close to the method used to define the spatial spectrum n_μ , the main difference being that, instead of computing the growth rates, we look at the nodes, i.e. at the spatial frequency or “average wavenumber” k of the Lyapunov vector.

3 THE GENERAL CASE

In this section we extend to time-dependent patterns the ideas sketched above for stationary random trajectories. More specifically, we shall consider orbits of temporal period T with $T > 1$. In this case, we have to solve the eigenvalue problem for the product matrix

$$U = \prod_{j=1}^T M_j D_\varepsilon \quad . \quad (5)$$

Notice that U is a banded matrix (of band width $2T + 1$) so that we are dealing with a sort of Schrödinger problem with long-range hopping. The fundamental difference is that not only U is not symmetric, but no similarity transformation can turn it into a symmetric matrix. This is confirmed by the generic existence of complex eigenvalues for $T > 2$ ¹.

It has to be admitted that this represents a serious mathematical difficulty, as the node theorem is rigorously proved (at least to our knowledge) only for a class of operators with a strictly real and positive spectrum [11]. Nevertheless, one can at least hope that some sort of ordering is maintained in the case of Lyapunov exponents. In fact, the latter are the logarithms of the eigenvalues of a matrix which is the the product of U by its transpose. Such a matrix is clearly symmetric and has a real and positive spectrum (we assume that no zero eigenvalues are present).

Nonetheless, even having accepted this optimistic point of view, one has to face further difficulties. The straightforward generalization of the method of the previous section would amount first to rewriting (1) as a spatial mapping. This requires the knowledge of the variables δx_n^i at all times in two consecutive sites and, accordingly, reads as

$$\delta Y^{i+1} = L^i(\Lambda) \delta Y^i \quad (6)$$

where δY is now a vector consisting of $2T$ components and $L^i(\Lambda)$ is the $2T \times 2T$ transfer matrix.

At variance with the case $T = 1$ discussed in the previous section, we cannot expect to get the correct value of n_λ by simply counting the nodes of one of the δx_n^i resulting from the repeated application of L^i to a randomly chosen initial vector δY^0 . In fact, the single eigenvector associated to a given eigenvalue (if we disregard the unlikely occurrence of degeneracies) corresponds to a unique trajectory of the $2T$ -dimensional transformation L^i (apart from an irrelevant scaling factor). Accordingly, it is very unlikely that a random choice of the initial conditions yields the right spatial structure, unless the number of nodes is independent of the trajectory, a possibility that must be ruled out on the basis of our numerical studies. This problem is very much reminiscent of the difficulty to determine the standard Lyapunov spectrum: in order to compute the m -th Lyapunov exponent, one cannot start from a randomly chosen initial condition: it is necessary to select m linearly independent vectors [5].

Notice that this very same problem would occur also in the safer case where node counting is known to apply, such as the Anderson model with long-range hopping or the harmonic chain with next-to-nearest neighbours coupling. Actually, an extension of the method based on counting the sign-changes of principal minors has been devised in the literature [12], but it seems definitely unpractical for matrices of large bandwidth (say for $T > 3$).

Having recognized such difficulties, and inspired by the analogy with the SOP, we have looked for a similar procedure in the present context, eventually finding an approach that works in all cases we have considered. As we have been unable to develop a rigorous proof, we present it here as a

Conjecture: The integrated density of Lyapunov exponents $n_\lambda(\lambda, 0)$ for a periodic orbit of period T coincides with the density of nodes along the T -th most expanding direction of the product of transfer matrices $L^i(e^\lambda)$.

In order to make the above conjecture really transparent, we need first to define the m -th expanding direction $Z_{(m)}^i$. We know that a generic vector $V_{(1)}^i$ aligns, after a suitable transient, along the most expanding direction $Z_{(1)}^i$, which can thus be easily identified. The concept of m -th most expanding direction is not, however, equally clear. In fact, we know that if we take m independent vectors $V_{(1)}^i, \dots, V_{(m)}^i$, the subspace $S^i(m)$ identified by the m vectors asymptotically orients itself as the most expanding m -dimensional subspace [5], but there is not a unique basis which identifies a given subspace $S^i(m)$: any set of linearly independent vectors generating $S^i(m)$ is equally meaningful. In fact, the SOP exploits this freedom to determine the vectors by imposing the mutual

¹For period-2 orbits it is still possible to reduce the matrix to a symmetric form with a suitable change of variables.

orthogonality, a condition motivated by the opportunity to minimize the numerical error. However, we have verified in several cases that this somehow arbitrary choice is not appropriate for the present purpose.

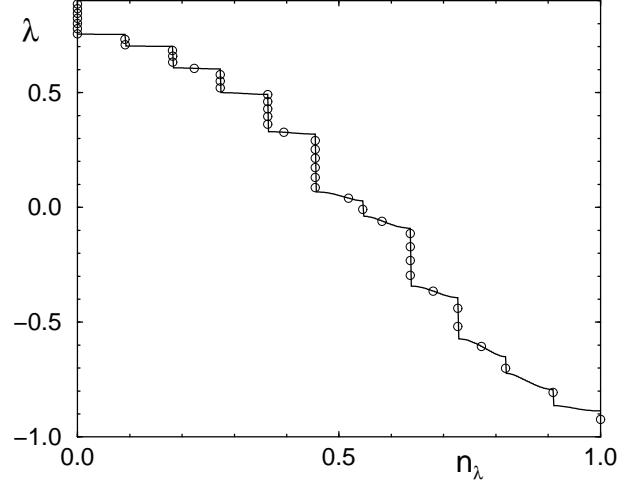


Figure 2: Lyapunov spectrum for an orbit of spatial period 11 and temporal period $T = 7$. The solid curve refers to the SOP, while open circles are the outcome of the node counting approach.

A meaningful solution can be found by realizing that $Z_{(m)}^i$ is the least expanding direction in the subspace $S^i(m)$. As a consequence, $Z_{(m)}^i$ is also the most expanding according to the backward evolution in $S^i(m)$. Since the most expanding direction of a given mapping is the only one which can be directly identified, we have finally an algorithm to determine $Z_{(m)}^i$, an algorithm that can be also taken as an operative definition of m -th most expanding direction. More precisely, one first iterates m vectors according to the general relation (6) in order to determine the sequence of subspaces $S^i(m)$ and the rules for the mapping of $S^i(m)$ onto $S^{i+1}(m)$. Afterwards, one must iterate a generic vector backward in space, restricting the dynamics to the sequence of subspaces $S^i(m)$. Notice that this restriction is very important, since any direction $Z_{(j)}^i$ with $j > m$ is more unstable than $Z_{(m)}^i$ so that any small but unavoidable numerical error would soon drive the trajectory towards the most unstable direction in the whole space R^{2T} .

Once we have been able to identify the m -th expanding direction, we can compute the nodes of δx_n^i along the T th direction and thus test the conjecture about $n_\lambda(\lambda, 0)$. In Figs. 2 and 3 we report two of the many examples we have studied to compare the Lyapunov spectrum determined through the SOP (solid curves) with the outcome of the node counting approach (open circles). In all cases we have found that the agreement is within the numerical accuracy.

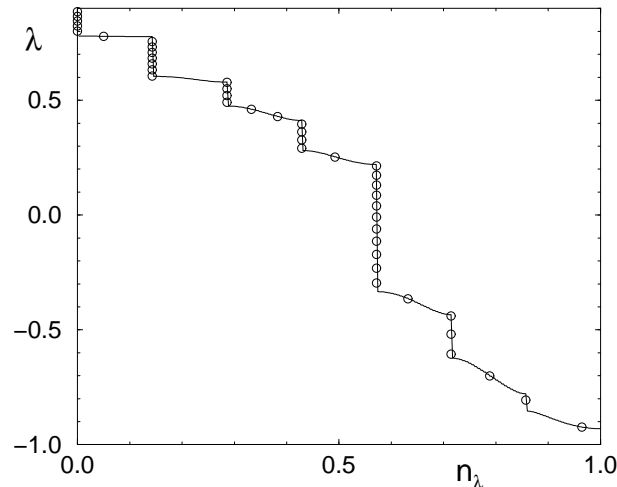


Figure 3: Same as in Fig. 2 for an orbit of spatial period 7 and temporal period $T = 5$.

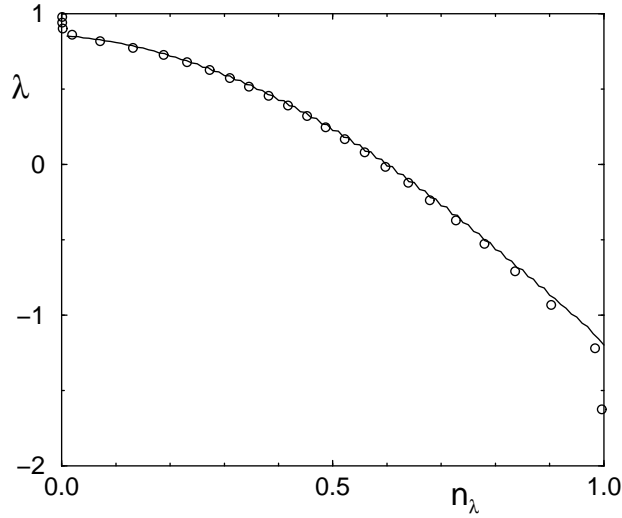


Figure 4: Lyapunov spectrum for a spatio-temporal random pattern. As for the above cases, open circles follow from the node counting while the solid curve correspond to the SOP.

It is important to stress that the conjecture appears to hold also in the fully chaotic regime, i.e. for random patterns both along the spatial and temporal direction. In this case, one has formally to consider temporal stripes of increasing height, i.e. to take the limit $T \rightarrow \infty$. The results reported in Fig. 4 for one such case hint at a general validity of the correspondence between node counting and Lyapunov spectra.

4 CONCLUSIONS AND PERSPECTIVES

In the previous section, we have seen that the spatial structure of the Lyapunov vector corresponding to the exponent λ contains the information necessary to determine the integrated density without the need to consider all Lyapunov exponents larger than λ , as required by the SOP. Although this may be considered as a computational advantage, we want to emphasize that the relevance of our conclusions, where proved to be rigorously true, does not come from the opportunity offered by the new algorithm. Indeed, we have seen that the study of the evolution along the spatial direction is not logically different from the application of the SOP used to follow the temporal evolution. The difference is that the size of the matrices involved in the evolution in the tangent space does not depend on the size of the system but on the periodicity of the solution. In the generic case of space-time chaos, it is a matter of the convergence rate versus space and time that makes one approach preferable to the other [13].

The relevance of our result relies on the possibility to attribute a meaning to the average spatial frequency of the Lyapunov vectors (the fraction of nodes). This confirms the intuition that not only the real but also the imaginary parts of the expansion rates $\tilde{\mu}$ are meaningful quantities and both contribute to the validity of the chronotopic approach. Let us indeed recall that the only cases in which we have been able to prove the existence of an entropy potential are those in which we have been able to interpret the spatial (and temporal) frequencies as suitable integrated densities.

The apparently coherent link between the node counting and the chronotopic approach is testified by the following extension of the conjecture in Sec. 3:

The integrated density of Lyapunov exponents $n_\lambda(\lambda, \mu)$ coincides, for T large enough, with the density of nodes along the m -th most expanding direction of the product of transfer matrices $L^i(e^\lambda)$, where m is fixed by the implicit condition $n_\mu(\lambda, \mu) = m/T$.

The above conjecture states that the nodes have to be counted along the direction characterized by the preassigned spatial growth rate μ . In practice, we fix λ (a free parameter in the transfer matrix expression) and iterate m vectors to determine the m -th most expanding direction. The rate

μ is the spatial Lyapunov exponent corresponding to the m -th direction, while n_λ is estimated by counting the corresponding nodes. The validity of the result can be tested by imposing a spatial profile equal to e^μ in the temporal evolution and thereby determine n_λ and λ with the SOP (this last value is the one reported in parenthesis in the table). In all cases we have considered, we have always found a good agreement. The data reported in the table refer to a uniform distribution of multipliers between 0 and 5.

A serious limitation to the validity of the results discussed in this paper is the restriction to positive multipliers (i.e. a positive slope of the local map, as for the generalized Bernoulli shift): all results have indeed been obtained for strictly positive m_n^i . However, we are confident that this limitation can be lifted and we are indeed working to clarify this crucial point.

Table 1: Comparison of the temporal Lyapunov spectrum $\lambda = \lambda(\mu)$ as determined with the node counting approach and the SOP (the value in parenthesis), for various combinations of the quantities n_μ , μ and n_λ . In the last column, the relative error (%) is reported.

n_μ	μ	n_λ	e^λ	error
0.0	1.711	0.005	4.000 (3.989)	0.2
0.5	2.635	0.602	4.000 (4.032)	0.8
0.9	3.122	0.924	4.000 (4.028)	0.7

More important, in our opinion, is the question whether the same approach can be extended to continuous-time and -space systems, i.e. to more realistic models of space-time chaos. We believe that instead of checking numerically whether this is true or not, it is more important to look for the possibly deep reasons that lie behind the apparent validity of the conjectures presented in this paper.

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References

- [1] S. Lepri, A. Politi and A. Torcini, J. Stat. Phys. **82** 1429 (1996).
- [2] S. Lepri, A. Politi and A. Torcini, J. Stat. Phys. **88** 31 (1997).
- [3] S. Lepri, A. Politi and A. Torcini, Chaos **7** 701 (1997).
- [4] D. Ruelle, Commun. Math. Phys. **87** 287 (1982).
- [5] I. Shimada and T. Nagashima, Prog. Theor. Phys. **61**, 1605 (1979); G. Benettin, L. Galgani, A. Giorgilli and J.M. Strelcyn, Meccanica, March 15 and 21 (1980).
- [6] R. Livi, G. Paladin, S. Ruffo and A. Vulpiani in *Advances in Phase Transitions and Disorder Phenomena*, eds G. Busiello *et al* (World Scientific, Singapore 1987).
- [7] I. Waller and R. Kapral, Phys. Rev. A **30** 2047 (1984); K. Kaneko, Prog. Theor. Phys. **72** 980 (1984).
- [8] S. Isola, A. Politi, S. Ruffo and A. Torcini, Phys. Lett. A **143** 365 (1990).
- [9] D.C. Mattis, *The Many-body Problem* (World Scientific, Singapore 1992).

- [10] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators* (Springer-Verlag, Berlin 1992) ch. III.
- [11] F.R. Gantmacher, *The Theory of Matrices* (Chelsea Publishing Company, New York 1959), see in particular the theorem at p. 105 vol. II.
- [12] J.L. Martin, Proc. Roy. Soc. **260**, 139 (1961).
- [13] A. Pikovsky and A. Politi, unpublished